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PRESSURE FLUCTUATIONS DUE TO THIN TURBULENT SHEAR LAYERS

By

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ABSTRACT

Integral formulas are derived for the pressure correlation and mean square fluctuating pressure $\langle p^2 \rangle_{TM}$ owing to the interaction of shear layers (i.e., planes of high mean shear separating regions of uniform, but different, mean velocity) with turbulence. Results for a single shear layer, assuming isotropic turbulence, indicate that in the far field (i.e., at distances from the shear layer large in comparison with the correlation length of the turbulence) $\langle p^2 \rangle_{TM}$ decays as the inverse fourth power of distance from the shear layer. Departure from isotropy, for a given mean kinetic energy of the turbulence, decreases the near field value of $\langle p^2 \rangle_{TM}$, but increases the far field value. For the double shear layer it is shown that if the distance between the shear layers is large in comparison with the correlation length of the turbulence, the resulting value of $\langle p^2 \rangle_{TM}$ is a superposition of the effects of the shear layers acting independently. Otherwise, interaction between the shear layers must be considered. These results are used to obtain order-of-magnitude estimates for the mean square wall pressure fluctuations in a turbulent boundary layer. These estimates show reasonably close agreement with other published results.

1.0 INTRODUCTION

The problem of pressure fluctuations in homogeneous, isotropic turbulence was first considered by Heisenberg[1] and Batchelor[2]. The effect of turbulence-mean flow interactions on turbulent pressure fluctuations was first studied by Kraichnan[3], who considered the interaction between a uniform mean shear and homogeneous, isotropic turbulence, and compared the results to the effects of "turbulence-turbulence" interaction. Subsequent investigations[4,5,6,7] have been concerned mainly with pressure fluctuations owing to boundary layer turbulence. The results of the work on boundary layer flows indicate that it is the turbulence-mean shear interaction which gives the major contribution to the pressure fluctuations.

In the theoretical analysis which follows, the problem of pressure fluctuations owing to the presence of shear layers in homogeneous turbulence is considered, with emphasis on the turbulence-mean shear contribution. The shear layers are taken to be planes of high mean shear which separate regions of uniform, but different, mean velocity. Flow conditions which can be approximated by one or more shear layers are found in the region immediately downstream from a rear-ward facing step in channel flow, the region near the exit of a jet, the immediate downstream area in the wake of a projectile, and in a boundary layer. Also, areas of strong wind shear separating adjacent regions of greatly differing wind velocity are frequently found in the atmosphere, and are usually associated with significant atmospheric turbulence.

In the following sections the formulas for the pressure correlation and the mean square fluctuating pressure owing to turbulence-mean shear interaction are derived, and are applied to the case of the single shear layer, and also to the case of the double shear layer. The effect of anisotropy of the turbulence is discussed. In the final section it is shown how the results obtained here can be used to give rough order-of-magnitude estimates of the wall pressure fluctuations in a turbulent boundary layer.

2.0 GENERAL THEORY - SINGLE SHEAR LAYER

The motion of an incompressible viscous fluid is governed by the Navier-Stokes equations:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = \nu \nabla^2 u_i \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 , \tag{2.2}$$

where u_i is the velocity, p the pressure, ρ the density, ν the coefficient of viscosity of the fluid, and x_i, t are the space and time variables, respectively. Here the indices i and j range over the values 1,2,3, and, in the usual tensor notation, a repeated index indicates a summation over that index. Differentiating Equation (2.1) with respect to x_i, and using (2.2), we obtain the equation for the pressure in terms of the velocity:

$$\nabla^2 p = -\rho \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j). \tag{2.3}$$

If we now write the pressure and velocity as the sum of mean and fluctuating parts, i.e.,

$$\begin{aligned} u_i &= U_i + \tilde{u}_i , \\ p &= P + \tilde{p} \end{aligned} \tag{2.4}$$

where

$$U_i = \langle u_i \rangle$$

$$P = \langle p \rangle$$

are the mean quantities (the brackets $\langle\ \rangle$ indicate an ensemble average or, since we are considering a process which is stationary in time, an equivalent time average). Substituting (2.4) into (2.3) we obtain

$$\nabla^2 P + \nabla^2 \tilde{p} = -\rho \frac{\partial^2}{\partial x_i \partial x_j}\left[U_i U_j + 2U_i \tilde{u}_j + \tilde{u}_i \tilde{u}_j\right]. \qquad (2.5)$$

Taking the mean of both sides of (2.5), and noting that the mean of any fluctuating quantity is zero, we obtain the equation for the mean pressure:

$$\nabla^2 P = -\rho \frac{\partial^2}{\partial x_i \partial x_j}\left[U_i U_j + \langle \tilde{u}_i \tilde{u}_j \rangle\right], \qquad (2.6)$$

where we have assumed that the process of differentiation and taking the mean commute. Subtracting (2.6) from (2.5) yields the equation for the fluctuating pressure:

$$\nabla^2 \tilde{p} = -\rho \frac{\partial^2}{\partial x_i \partial x_j}\left[2U_i \tilde{u}_j + \tilde{u}_i \tilde{u}_j - \langle \tilde{u}_i \tilde{u}_j \rangle\right]. \qquad (2.7)$$

We now drop the tilde notation, and for the remainder of the paper the symbols p and u_i will denote fluctuating pressure and velocity, respectively. Equation (2.7) then becomes

$$\nabla^2 p = -\rho \frac{\partial^2}{\partial x_i \partial x_j}\left[2U_i u_j + u_i u_j - \langle u_i u_j \rangle\right] . \tag{2.8}$$

Now the equation

$$\nabla^2 p = -\rho f(\vec{x}),$$

where $f(\vec{x})$ decays to zero sufficiently rapidly at ∞, has the solution

$$p(\vec{x}) = \frac{\rho}{4\pi} \int \frac{f(\vec{y})}{|\vec{x} - \vec{y}|} d\vec{y} ,$$

where $d\vec{y} = dy_1 \, dy_2 \, dy_3$ and the integral is taken over all space. (Here, and in the notation which follows, a symbol denoting a vector quantity is written with the arrow above, while the same symbol without the arrow denotes the length of the vector, e.g.,

$$\vec{x} = (x_1, x_2, x_3)$$

$$x = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} .$$

Also, sometimes $(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}$ is written as $|\vec{x}|$.) Then, assuming the turbulent velocity is zero outside some finite region, the solution to (2.8) is

$$p(\vec{x},t) = \frac{\rho}{4\pi}\int \frac{\partial^2}{\partial y_i \partial y_j}\left[2U_i(\vec{y})u_j(\vec{y},t) + u_i(\vec{y},t)u_j(\vec{y},t) - \langle u_i(\vec{y},t)u_j(\vec{y},t)\rangle\right] \cdot$$

$$\left|\vec{x}-\vec{y}\right|^{-1} d\vec{y} \, . \qquad (2.9)$$

It follows from (2.9) that

$$p(\vec{x},t)\, p(\vec{x}\,',t') = \frac{\rho^2}{16\pi^2}\iint \frac{\partial^2}{\partial y_i \partial y_j}\left[2U_i u_j + u_i u_j - \langle u_i u_j\rangle\right] \cdot$$

$$\frac{\partial^2}{\partial y_k' \partial y_\ell'}\left[2U_k' u_\ell' + u_k' u_\ell' - \langle u_k' u_\ell'\rangle\right]\left|\vec{x}-\vec{y}\right|^{-1}\left|\vec{x}\,'-\vec{y}\,'\right|^{-1} d\vec{y}\, d\vec{y}\,' , \qquad (2.10)$$

where $u_i = u_i(\vec{y},t)$, $u_i' = u_i(\vec{y}\,',t')$, etc., and the indices k, ℓ range over the values 1, 2, 3. Taking the mean of both sides of (2.10), and neglecting third order means (see Batchelor[2] for a discussion of this point) we obtain

$$\langle p(\vec{x},t)\, p(\vec{x}\,',t')\rangle = \frac{\rho^2}{16\pi^2}\iint \Bigg\{ 4 \left\langle \frac{\partial^2}{\partial y_i \partial y_j}(U_i u_j) \frac{\partial^2}{\partial y_k' \partial y_\ell'}(U_k' u_\ell')\right\rangle +$$

$$\left\langle \frac{\partial^2}{\partial y_i \partial y_j}(u_i u_j) \frac{\partial^2}{\partial y_k' \partial y_\ell'}(u_k' u_\ell')\right\rangle - \frac{\partial^2}{\partial y_i \partial y_j}\langle u_i u_j\rangle \frac{\partial^2}{\partial y_k' \partial y_\ell'}\langle u_k' u_\ell'\rangle \Bigg\} \cdot$$

$$\left|\vec{x}-\vec{y}\right|^{-1}\left|\vec{x}\,'-\vec{y}\,'\right|^{-1} d\vec{y}\, d\vec{y}\,' . \qquad (2.11)$$

Utilizing the fact that the mean and fluctuating velocities each satisfy the continuity equation, and taking the differentiation outside the brackets, (2.11) can be written

$$\langle p(\vec{x},t)\, p(\vec{x}',t')\rangle = \frac{\rho^2}{16\pi^2}\iint \Bigg\{ 4\, \frac{\partial U_i}{\partial y_j}\, \frac{\partial U_k'}{\partial y_\ell'}\, \frac{\partial^2}{\partial y_i \partial y_k'} \langle u_j u_\ell' \rangle + \frac{\partial^4}{\partial y_i \partial y_j \partial y_k' \partial y_\ell'} \Big(\langle u_i u_j u_k' u_\ell' \rangle - \langle u_i u_j \rangle \langle u_k' u_\ell' \rangle \Big) \Bigg\} \cdot |\vec{x}-\vec{y}|^{-1}\, |\vec{x}'-\vec{y}'|^{-1}\, d\vec{y}\, d\vec{y}'. \tag{2.12}$$

The first term in the integrand on the right-hand side of (2.12) represents the contribution to the pressure correlation due to the interaction of the turbulence with the mean flow, while the second term gives the contribution due to the turbulence-turbulence interaction. Denoting these terms by $\langle p(\vec{x},t)p(\vec{x}',t')\rangle_{TM}$ and $\langle p(\vec{x},t)\, p(\vec{x}',t')\rangle_{TT}$, respectively, we have

$$\langle p(\vec{x},t)\, p(\vec{x}',t')\rangle = \langle p(\vec{x},t)\, p(\vec{x}',t')\rangle_{TM} + \langle p(\vec{x},t)\, p(\vec{x}',t')\rangle_{TT},$$

where

$$\langle p(\vec{x},t)\, p(\vec{x}',t')\rangle_{TM} =$$

$$\frac{\rho^2}{4\pi^2}\iint \frac{\partial U_i}{\partial y_j}\, \frac{\partial U_k'}{\partial y_\ell'} \left(\frac{\partial^2}{\partial y_i \partial y_k'} \langle u_j u_\ell' \rangle \right) |\vec{x}-\vec{y}|^{-1}\, |\vec{x}'-\vec{y}'|^{-1}\, d\vec{y}\, d\vec{y}'; \tag{2.13}$$

and

$$\langle p(\vec{x},t)\, p(\vec{x}',t')\rangle_{TT} =$$

$$\frac{\rho^2}{16\pi^2}\iint \left\{ \frac{\partial^4}{\partial y_i \partial y_j \partial y_k' \partial y_\ell'} \left(\langle u_i u_j u_k' u_\ell' \rangle - \langle u_i u_j \rangle \langle u_k' u_\ell' \rangle \right) \right\} \cdot$$

$$|\vec{x}-\vec{y}|^{-1}\, |\vec{x}'-\vec{y}'|^{-1} \quad d\vec{y}\, d\vec{y}' \,. \tag{2.14}$$

The turbulence-turbulence term (2.14) has been examined in detail by Kraichnan[3] for the case of isotropic turbulence. Kraichnan, in the same paper, also considered the turbulence-mean flow term (2.13) for the case of a uniform mean shear superimposed on isotropic turbulence. In the present work we shall consider the turbulence-mean flow term when the mean shear, instead of being uniform, is confined to one or more planes in three-dimensional space, these planes separating regions of different, but uniform, mean velocity. These planes are then the shear planes, or shear layers.

We consider first the case of the single shear layer, which we assume to be coincident with the plane $x_3 = 0$, and which separates semi-infinite regions of uniform mean velocity V_0 and V_1, respectively, in the x_1 direction. The mean velocity profile for this case can then be written

$$U_1(x_3) = V_0 + (V_1 - V_0)\, H(x_3) \,,$$

where H denotes the Heaviside unitary function, i.e.,

$$H(x_3) = \begin{cases} 0 & \text{for } x_3 < 1 \\ 1 & \text{for } x_3 > 1 \end{cases} .$$

Differentiating with respect to x_3 we obtain the mean shear:

$$\frac{\partial U_1}{\partial x_3} = (V_1 - V_0)\,\delta(x_3), \tag{2.15}$$

where $\delta(x_3)$ refers to the Dirac delta function. Substituting (2.15) into (2.13) and integrating with respect to the x_3 and x_3' variables, we obtain the integral formula for the pressure correlation owing to the interaction of the turbulence with the shear layer:

$$\langle p(\vec{x})\,p(\vec{x}')\rangle_{TM} = \frac{\rho^2}{4\pi^2}(V_1 - V_0)^2 \iint \frac{\partial^2 R_{33}}{\partial\eta_1\,\partial\eta_1'}(\vec{\eta},\vec{\eta}')\left|\vec{x}-\vec{\eta}\right|^{-1}\left|\vec{x}'-\vec{\eta}'\right|^{-1} \cdot d\vec{\eta}\,d\vec{\eta}' \tag{2.16}$$

where $R_{ij}(y,y')$ denotes the velocity correlation $\langle u_i(y)\,u_j(y')\rangle$, $\vec{\eta} = (y_1, y_2, 0)$, $\vec{\eta}' = (y_1', y_2', 0)$, and $d\vec{\eta} = d\eta_1 d\eta_2$. (We have here suppressed the time variable by setting $t = t'$.) The right-hand side of (2.16) may be integrated by parts to yield

$$\langle p(\vec{x})\,p(\vec{x}')\rangle_{TM} = \frac{\rho^2}{4\pi^2} V^2 \iint R_{33}(\vec{\eta},\vec{\eta}')\,\frac{\partial}{\partial\eta_1}\left|\vec{x}-\vec{\eta}\right|^{-1}\frac{\partial}{\partial\eta_1'}\left|\vec{x}'-\vec{\eta}'\right|^{-1} d\vec{\eta}\,d\vec{\eta}' \tag{2.17}$$

where we have here set $V \equiv |V_1 - V_0|$.

If the turbulence is homogeneous (i.e., $R_{33} = R_{33}(\vec{\eta} - \vec{\eta}\,')$), (2.17) can be written

$$\langle p(\vec{x})\, p(\vec{x}\,')\rangle_{TM} = \frac{\rho^2}{4\pi^2} V^2 \int \frac{\partial}{\partial x_1'} |\vec{x}\,' - \vec{\eta}\,'|^{-1} \int R_{33}(\vec{\eta} - \vec{\eta}\,') \frac{\partial}{\partial x_1} |\vec{x} - \vec{\eta}|^{-1} d\vec{\eta}\, d\vec{\eta}\,',$$

or, letting $\vec{\zeta} = \vec{\eta} - \vec{\eta}\,'$,

$$\langle p(\vec{x})\, p(\vec{x}\,')\rangle_{TM} = \frac{\rho^2}{4\pi^2} V^2 \int \frac{\partial}{\partial x_1'} |\vec{x}\,' - \vec{\eta}\,'|^{-1} \int R_{33}(\vec{\zeta}) \frac{\partial}{\partial x_1} |\vec{x} - \vec{\zeta} - \vec{\eta}\,'|^{-1} d\vec{\zeta}\, d\vec{\eta}\,' =$$

$$\frac{\rho^2}{4\pi^2} V^2 \iint R_{33}(\vec{\zeta}) \frac{\partial}{\partial x_1} |\vec{x} - \vec{\zeta} - \vec{\eta}\,'|^{-1} \frac{\partial}{\partial x_1'} |\vec{x}\,' - \vec{\eta}\,'|^{-1} d\vec{\zeta}\, d\vec{\eta}\,' =$$

$$\frac{\rho^2}{4\pi^2} V^2 \int R_{33}(\vec{\zeta}) \int \frac{\partial}{\partial x_1} |\vec{x} - \vec{\zeta} - \vec{\eta}\,'|^{-1} \frac{\partial}{\partial x_1'} |\vec{x}\,' - \vec{\eta}\,'|^{-1} d\vec{\eta}\,' d\vec{\zeta}. \qquad (2.18)$$

Now

$$\frac{\partial}{\partial x_1} |\vec{x} - \vec{\zeta} - \vec{\eta}\,'|^{-1} \frac{\partial}{\partial x_1'} |\vec{x}\,' - \vec{\eta}\,'|^{-1} = \frac{x_1 - \zeta_1 - \eta_1'}{|\vec{x} - \vec{\zeta} - \vec{\eta}\,'|^3} \; \frac{x_1' - \eta_1'}{|\vec{x}\,' - \vec{\eta}\,'|^3} \quad ,$$

so that (2.18) can be written, after setting $\vec{\xi} \equiv \vec{\eta}\,'$,

$$\langle p(\vec{x})\, p(\vec{x}\,')\rangle_{TM} = \frac{\rho^2}{4\pi^2} V^2 \int R_{33}(\vec{\zeta}) \int \frac{x_1 - \zeta_1 - \xi_1}{|\vec{x} - \vec{\zeta} - \vec{\xi}|^3} \; \frac{x_1' - \xi_1}{|\vec{x}\,' - \vec{\xi}|^3} \, d\vec{\xi}\, d\vec{\zeta}. \qquad (2.19)$$

Setting $\vec{x} = \vec{x}' = (0,0,h)$ in (2.19) we obtain the formula for the mean square fluctuating pressure due to turbulence-mean shear interaction at a distance h from the shear layer, which we denote by $\langle p^2(h)\rangle_{TM}$, in the form

$$\langle p^2(h)\rangle_{TM} = \frac{\rho^2}{4\pi^2} V^2 \int R_{33}(\vec{\zeta}) \int \frac{\xi_1}{(h^2+\xi^2)^{3/2}} \frac{\xi_1+\zeta_1}{(h^2+|\vec{\xi}+\vec{\zeta}|^2)^{3/2}} \, d\vec{\xi}\, d\vec{\zeta}. \tag{2.20}$$

The $\vec{\xi}$ integration of Equation (2.20) is carried out in Appendix A. Substituting that result (Equation (A16)) into (2.20) we obtain

$$\langle p^2(h)\rangle_{TM} = (2\pi)^{-1} \rho^2 V^2 \int \zeta^{-2} R_{33}(\vec{\zeta}) \left\{ (1 - 2\zeta_1^2/\zeta^2)\left[1-(1+\zeta^2/4h^2)^{-\frac{1}{2}}\right] + (\zeta_1^2/4h^2)(1+\zeta^2/4h^2)^{-3/2} \right\} d\vec{\zeta}. \tag{2.21}$$

We now apply the above results to the case when the turbulence is isotropic. The velocity correlation for this case can be written, following Batchelor[8],

$$R_{ij}(\vec{r}) = v^2 \left[\frac{1}{2} r(\delta_{ij} - r_i r_j/r^2) f'(r) + \delta_{ij} f(r)\right],$$

where v^2 is the mean square of any turbulent velocity component, $f(r)$ is the longitudinal velocity correlation coefficient for the turbulence, and δ_{ij} is the Kronecker delta. Then

$$R_{33}(\vec{\zeta}) = R_{33}(\zeta_1, \zeta_2, 0) = v^2\left[f(\zeta) + \frac{1}{2}\zeta f'(\zeta)\right]. \tag{2.22}$$

Substituting (2.22) into (2.21) and letting ϕ denote the angle between the vector (ζ_1, ζ_2) and the ζ_1 axis, we have

$$J(z) = (2\pi)^{-1} \rho^2 V^2 v^2 \int_0^{\infty}\int_0^{2\pi} \zeta^{-2} \left[f(\zeta) + \frac{1}{2}\zeta f'(\zeta) \right] \Big\{ (1 - 2\cos^2\phi) \cdot$$

$$\left[1 - (1 + \zeta^2/z^2)^{-\frac{1}{2}} \right] + (\zeta^2/z^2)(1 + \zeta^2/z^2)^{-3/2} \cos^2\phi \Big\} \zeta \, d\phi \, d\zeta, \tag{2.23}$$

where we have, for convenience, introduced the notation

$z = 2h$ and $J(z) = \langle p^2(h) \rangle_{TM} = \langle p^2(z/2) \rangle_{TM}$. Integrating first with respect to ϕ gives

$$J(z) = \frac{1}{2} \rho^2 v^2 V^2 z^{-2} \int_0^{\infty} (1 + \zeta^2/z^2)^{-3/2} \left[\zeta f(\zeta) + \frac{1}{2} \zeta^2 f'(\zeta) \right] d\zeta . \tag{2.24}$$

Noting that

$$\zeta f(\zeta) + \frac{1}{2} \zeta^2 f'(\zeta) = \frac{d}{d\zeta} \left[\frac{1}{2} \zeta^2 f(\zeta) \right],$$

the right hand side of Equation (2.24) can be integrated by parts to yield

$$J(z) = \frac{3}{4} \rho^2 v^2 V^2 z^{-4} \int_0^{\infty} \zeta^3 (1 + \zeta^2/z^2)^{-5/2} f(\zeta) \, d\zeta, \tag{2.25}$$

which can be re-written in the form

$$J(z) = \frac{3}{4}\rho^2 v^2 V^2 z \int_0^\infty \zeta^3 (\zeta^2 + z^2)^{-5/2} f(\zeta)\, d\zeta \,. \tag{2.26}$$

We define

$$I(z) = z \int_0^\infty \zeta^3 (\zeta^2 + z^2)^{-5/2} f(\zeta)\, d\zeta \,, \tag{2.27}$$

so that $I(z)$ is essentially a normalized mean-square pressure fluctuation. Making the change of variable $\zeta = z\eta$, (2.27) can be written

$$I(z) = \int_0^\infty \eta^3 (1 + \eta^2)^{-5/2} f(z\eta)\, d\eta \,; \; z > 0. \tag{2.28}$$

Differentiating (2.28) with respect to z we obtain the formulas

$$I'(z) = \int_0^\infty \eta^4 (1 + \eta^2)^{-5/2} f'(z\eta)\, d\eta \; ; z > 0, \tag{2.29}$$

$$I''(z) = \int_0^\infty \eta^5 (1 + \eta^2)^{-5/2} f''(z\eta)\, d\eta \; ; z > 0 \,. \tag{2.30}$$

Since $f(\zeta)$ is a non-increasing function of ζ we must have $f'(\zeta) \leq 0$ for all ζ, so that, by (2.29), $I'(z) < 0$ for $z > 0$. Integrating by parts in (2.30) we have

$$I''(z) = -5z^{-1}\int_0^{\infty} \eta^4 (1+\eta^2)^{-7/2} f'(z\eta)\,d\eta\ ;\ z>0,$$

so that $I''(z) > 0$ for $z>0$. Also, letting $z \to 0$ in (2.28), $f(z\eta) \to f(0) = 1$ for all $\eta > 0$, so that, carrying out the integration, we obtain $I(0) = \frac{2}{3}$; i.e., since $J(z) = \frac{3}{4}\rho^2 v^2 V^2 I(z)$,

$$\langle p^2(0)\rangle_{TM} = \frac{1}{2}\rho^2 V^2 v^2 \quad . \tag{2.31}$$

Note that we cannot obtain $I'(0)$ and $I''(0)$ by a similar limiting process, since the functions $\eta^4(1+\eta^2)^{-5/2}$ and $\eta^5(1+\eta^2)^{-5/2}$ are not integrable on the interval $0 \le \eta < +\infty$; however, it can be shown, assuming $f'(0) = 0$, that $I'(z)$ and $I''(z)$ approach finite limits as $z \to 0$. Also, it is clear, by writing $I(z)$ in the form

$$I(z) = z^{-4}\int_0^{\infty} \zeta^3 (1 + \zeta^2/z^2)^{-5/2} f(\zeta)\,d\zeta, \tag{2.32}$$

that $I(z)$ decays like z^{-4} as $z \to \infty$; i.e., in the far field, the mean square pressure fluctuation due to turbulence-mean shear interaction decays as the inverse fourth power of distance from the shear layer. (The near and far fields of the shear layer are defined according to whether the distance to the shear layer is much less than, or much greater than, the correlation length of the turbulence. This, of course, is not to be confused with the acoustic near and f . fields, since we are dealing here only with incompressible flows.) We can then sketch, $I(z)$ or, equivalently, $\langle p^2(h)\rangle_{TM}$, as shown in Figure 1.

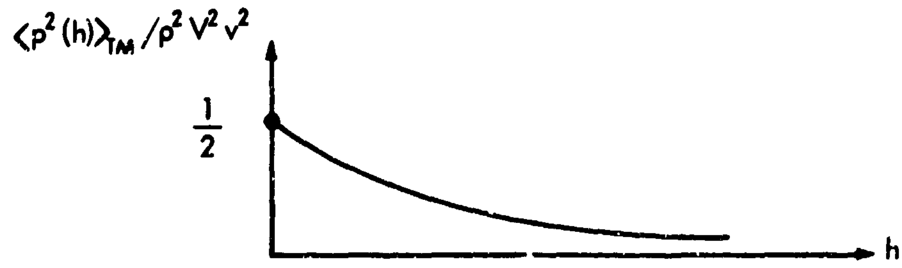

Fig. 1. Normalized mean-square pressure fluctuation due to turbulence-mean shear interaction vs. distance from shear layer (isotropic turbulence).

We can obtain an asymptotic expression for $\langle p^2(h)\rangle_{TM}$ for large h as follows: We first take $f(\zeta) = e^{-\sigma^2\zeta^2}$, where σ^{-1} is the correlation length of the turbulence. Substituting this into (2.32) we obtain

$$I(z) = z^{-4}\int_0^\infty \zeta^3(1+\zeta^2/z^2)^{-5/2}\, e^{-\sigma^2\zeta^2}\, d\zeta ,$$

which becomes, after a change of variable,

$$I(z) = \frac{1}{2}(\sigma z)^{-4}\int_0^\infty u(1+u/\sigma^2 z^2)^{-5/2}\, e^{-u}\, du . \qquad (2.33)$$

Now

$$\int_0^\infty u(1+u/\sigma^2 z^2)^{-5/2}\, e^{-u}\, du = \int_0^{\sigma^2 z^2} u(1+u/\sigma^2 z^2)^{-5/2}\, e^{-u}\, du +$$

$$\int_{\sigma^2 z^2}^\infty u(1+u/\sigma^2 z^2)^{-5/2}\, e^{-u}\, du . \qquad (2.34)$$

Then, using a power series expansion, we can write

$$\int_0^{\sigma^2 z^2} u(1 + u/\sigma^2 z^2)^{-5/2} e^{-u}\, du = \int_0^{\sigma^2 z^2} u\left(1 - \frac{5}{2} u/\sigma^2 z^2 + O(\sigma^{-4} z^{-4})\right) e^{-u}\, du$$

$$= 1 + \left\{ \text{higher order terms in } (\sigma z)^{-1} \right\}.$$

It can easily be shown that the second term on the right hand side of (2.34) involves only higher order terms in $(\sigma z)^{-1}$, so that, for large σz,

$$I(z) \cong \frac{1}{2}(\sigma z)^{-4}. \tag{2.35}$$

Then since

$$\langle p^2(h)\rangle_{TM} = J(z) = \frac{3}{4} \rho^2 V^2 v^2 \; I(z) ,$$

we can write, after substituting for $z = 2h$,

$$\langle p^2(h)\rangle_{TM} \cong \frac{3}{128} \rho^2 V^2 v^2 (\sigma h)^{-4} \tag{2.36}$$

for σh large; i.e., for h large in comparison with the correlation length of the turbulence.

3.0 TURBULENCE WITH SCALE ANISOTROPY

We consider now the somewhat more realistic situation in which the turbulent eddies are elongated in the direction of the mean flow. Following Kraichnan[3], we introduce the anisotropic velocity correlation function

$$R_{ij}^{(a)}(\zeta_1, \zeta_2, \zeta_3) = 3(2+\alpha^2)^{-1} \alpha^{\delta_{i1}} \alpha^{\delta_{j1}} R_{ij}(\alpha^{-1}\zeta_1, \zeta_2, \zeta_3), \tag{3.1}$$

where $\alpha > 1$ is the anisotropic scale factor and $R_{ij}(\vec{\zeta})$ corresponds to isotropic turbulence. It can easily be shown that $R_{ij}^{(a)}(\vec{\zeta})$ satisfies $\dfrac{\partial R_{ij}^{(a)}(\vec{\zeta})}{\partial \zeta_j} = 0$; also that $R_{ij}^{(a)}(\vec{\zeta})$ corresponds to a turbulent flow in which the mean kinetic energy density is the same as that for isotropic case, and also a flow in which the correlation length is stretched by a factor α in the ζ_1 direction.

Now since $R_{ij}(\vec{\zeta})$ corresponds to isotropic turbulence, we can write

$$R_{ij}(\vec{\zeta}) = v^2\left[(\delta_{ij} - \zeta_i\zeta_j/\zeta^2)\frac{1}{2}\zeta f'(\zeta) + \delta_{ij} f(\zeta)\right]. \tag{3.2}$$

Then substituting Equation (3.2) into (3.1) we obtain

$$R_{33}^{(a)}(\zeta_1, \zeta_2, 0) = 3(2+\alpha^2)^{-1} v^2 \left[\frac{1}{2}(\zeta^2 - \gamma^2\zeta_1^2)^{\frac{1}{2}} f'\left((\zeta^2 - \gamma^2\zeta_1^2)^{\frac{1}{2}}\right) + f\left((\zeta^2 - \gamma^2\zeta_1^2)^{\frac{1}{2}}\right)\right], \tag{3.3}$$

where $\gamma^2 = 1 - \alpha^{-2}$, and $\zeta^2 = \zeta_1^2 + \zeta_2^2$. Letting ϕ be the angle between the vector (ζ_1, ζ_2) and the ζ_1 axis, and defining $\lambda = (1 - \gamma^2 \cos^2\phi)^{\frac{1}{2}}$, Equation (3.3) can be written

$$R_{33}^{(a)}(\zeta_1, \zeta_2, 0) = 3(2 + \alpha^2)^{-1} v^2 \left[f(\lambda\zeta) + \frac{1}{2}\lambda\zeta f'(\lambda\zeta) \right]. \tag{3.4}$$

Substituting (3.4) into (2.21), and setting $z = 2h$ as before, we obtain

$$J^{(a)}(z) = \frac{3\rho^2 v^2 V^2}{2\pi(2+\alpha^2)} \int_0^\infty \int_0^{2\pi} \left[f(\lambda\zeta) + \frac{1}{2}\lambda\zeta f'(\lambda\zeta) \right] \Big\{ (1 - 2\cos^2\phi) \cdot$$

$$\left[1 - (1 + \zeta^2/z^2)^{-\frac{1}{2}} \right] + (\zeta^2/z^2)(1 + \zeta^2/z^2)^{-3/2} \cos^2\phi \Big\} \zeta^{-1} d\phi\, d\zeta, \tag{3.5}$$

where $J^{(a)}(z)$ is the anisotropic counterpart of $J(z)$. Noting that

$$\zeta f(\lambda\zeta) + \frac{1}{2}\lambda\zeta^2 f'(\lambda\zeta) = \frac{d}{d\zeta}\left[\frac{1}{2}\zeta^2 f(\lambda\zeta) \right],$$

we can use integration by parts on Equation (3.5) to obtain

$$J^{(a)}(z) = \frac{3\rho^2 v^2 V^2}{2\pi(2+\alpha^2)} \int_0^{2\pi} \int_0^\infty (1 - 2\cos^2\phi) \Big\{ \left[1 - (1 + \zeta^2/z^2)^{-\frac{1}{2}} \right] -$$

$$(\zeta^2/2z^2)(1 + \zeta^2/z^2)^{-3/2} \Big\} \zeta^{-1} f(\lambda\zeta)\, d\zeta\, d\phi \; +$$

$$\frac{9\rho^2 v^2 V^2}{4\pi(2+\alpha^2)z^4}\int_0^{2\pi}\int_0^{\infty} \cos^2\phi\,(1+\zeta^2/z^2)^{-5/2}\,\zeta^3 f(\lambda\zeta)\,d\zeta\,d\phi \;. \qquad (3.6)$$

Making the change of variable $\eta = \zeta/z$, Equation (3.6) becomes

$$J^{(a)}(z) = \frac{3\rho^2 V^2 v^2}{2\pi(2+\alpha^2)}\int_0^{2\pi}\int_0^{\infty}(1-2\cos^2\phi)\left[1-(1+\eta^2)^{-\frac{1}{2}}-\frac{1}{2}\eta^2(1+\eta^2)^{-3/2}\right]\cdot$$

$$\eta^{-1} f(\lambda z\eta)\,d\eta\,d\phi \;+$$

$$\frac{9\rho^2 V^2 v^2}{4\pi(2+\alpha^2)}\int_0^{2\pi}\cos^2\phi\left[\eta^3(1+\eta^2)^{-5/2}\right]f(\lambda z\eta)\,d\eta\,d\phi \;. \qquad (3.7)$$

Letting $z \to 0$, in (3.7) we obtain

$$J^{(a)}(0) = \frac{3\rho^2 V^2 v^2}{2(2+\alpha^2)} \;,$$

or, letting $\langle p^2(h)\rangle^{(a)}_{TM}$ denote the mean square fluctuating pressure at a distance h from the shear layer for the anisotropic case, we have

$$\langle p^2(0)\rangle^{(a)}_{TM} = \frac{3}{2+\alpha^2}\quad\frac{1}{2}\,\rho^2 V^2 v^2 \;. \qquad (3.8)$$

Upon comparing this result with (2.31) we see that anisotropy results in a reduction of mean square fluctuating pressure at the shear layer.

In order to obtain far field results for $J^{(a)}(z)$ we first assume

$$f(\zeta) = e^{-\sigma^2 \zeta^2} ,$$

so that

$$f(\lambda\zeta) = e^{-\sigma^2 \lambda^2 \zeta^2} = e^{-\sigma^2 (1 - \gamma^2 \cos^2 \phi) \zeta^2} .$$

Using a procedure similar to one used previously (see Equations (2.33)-(2.35)), the integrands of Equation (3.6) are expanded in powers of ζ/z and terms of order $(1/\sigma z)^6$ and higher are discarded. Then after integrating with respect to ζ, (3.6) becomes

$$J^{(a)}(z) \cong \frac{3}{8} \frac{\rho^2 v^2 V^2}{\sigma^4 z^4} \frac{3}{2+\alpha^2} \frac{1}{4\pi} \int_0^{2\pi} \frac{1 + 2\cos^2 \phi}{(1-\gamma^2 \cos^2 \phi)^2} d\phi . \tag{3.9}$$

The integral on the right hand side of Equation (3.9) can be integrated by means of tables, with the result

$$J^{(a)}(z) \cong \frac{3}{8} \frac{\rho^2 v^2 V^2}{\sigma^4 z^4} K(\alpha) , \tag{3.10}$$

for σz large, where

$$K(\alpha) = \frac{3\alpha(1+3\alpha^2)}{4(2+\alpha^2)} . \tag{3.11}$$

Since

$$J^{(a)}(z) = \langle p^2(h)\rangle_{TM}^{(a)}$$

Equation (3.10) can be put into the alternate form

$$\langle p^2(h)\rangle_{TM}^{(a)} \cong \frac{3}{128} \frac{\rho^2 v^2 V^2}{(\sigma h)^4} K(\alpha), \tag{3.12}$$

for large σh, where h is the distance from the shear layer. This is to be compared to the isotropic result (Equation (2.36)).

Note that $K(\alpha)$ is monotonically increasing with α, i.e., for mean kinetic energy density fixed, the mean square pressure fluctuations due to turbulence-mean flow interaction at large distances from the shear layer increases with increasing anisotropy of the turbulence.

4.0 DOUBLE SHEAR LAYER

It is appropriate here to consider the case of the double shear layer, since, as will be seen, the effect of multiple shear layers on the mean square pressure fluctuations is not a simple superposition of the effects of the individual shear layers. Also, the results obtained here will be of use in the discussion of boundary layer turbulence.

The symmetric double shear layer considered here is characterized by a region, say $|x_3| \leq a$ in $x_1 - x_2 - x_3$ space, in which the mean velocity is uniform and is equal to V_0 in the x_1 direction. This region separates two semi-infinite regions in which the mean velocity is uniform and is equal to V_1 in the x_1 direction. The planes $x_3 = \pm a$ are then the shear planes.

The procedure is similar to that of Section 2. We first write the velocity profile in the form

$$U_1(x_3) = V_1 + (V_0 - V_1)\, H(x_3 + a) + (V_1 - V_0)\, H(x_3 - a).$$

Differentiating, we obtain the mean shear:

$$\frac{\partial U_1}{\partial x_3} = (V_1 - V_0)\left[\delta(x_3 - a) - \delta(x_3 + a)\right] \cdot$$

Substituting the expression for the mean shear into Equation (2.13) we obtain

$$\langle p(\vec{x})p(\vec{x}')\rangle_{TM} = \frac{\rho^2 V^2}{4\pi^2}\iint \left\{ \frac{\partial^2 R_{33}}{\partial\eta_1 \partial\eta_1'}(\vec{\eta}_+, \vec{\eta}_+')|\vec{x} - \vec{\eta}_+|^{-1}|\vec{x}' - \vec{\eta}_+'|^{-1} - \right.$$

$$\frac{\partial^2 R_{33}}{\partial\eta_1 \partial\eta_1'}(\vec{\eta}_+, \vec{\eta}_-')|\vec{x} - \vec{\eta}_+|^{-1}|\vec{x}' - \vec{\eta}_-'|^{-1} -$$

$$\frac{\partial^2 R_{33}}{\partial\eta_1 \partial\eta_1'}(\vec{\eta}_-, \vec{\eta}_+')|\vec{x} - \vec{\eta}_-|^{-1}|\vec{x}' - \vec{\eta}_+'|^{-1} +$$

$$\left. \frac{\partial^2 R_{33}}{\partial\eta_1 \partial\eta_1'}(\vec{\eta}_-, \vec{\eta}_-')|\vec{x} - \vec{\eta}_-|^{-1}|\vec{x}' - \vec{\eta}_-'|^{-1} \right\} d\vec{\eta}\, d\vec{\eta}' , \qquad (4.1)$$

where

$$\vec{\eta} = (\eta_1, \eta_2) ,$$
$$\vec{\eta}_+ = (\eta_1, \eta_2, a),$$
$$\vec{\eta}_- = (\eta_1, \eta_2, -a),$$
$$\vec{\eta}' = (\eta_1', \eta_2'),$$
$$\vec{\eta}_+' = (\eta_1', \eta_2', a),$$
$$\vec{\eta}_-' = (\eta_1', \eta_2' -a),$$

and $V = |V_0 - V_1|$ as before. Integrating the right hand side of (4.1) by parts yields

$$\langle p(\vec{x}), (\vec{x}')\rangle_{TM} = \frac{\rho^2 V^2}{4\pi^2} \iint \Big\{ R_{33}(\vec{\eta}_+, \vec{\eta}'_+) \frac{\partial}{\partial \eta_1} |\vec{x} - \vec{\eta}_+|^{-1} \frac{\partial}{\partial \eta'_1} |\vec{x}' - \vec{\eta}'_+|^{-1} -$$

$$R_{33}(\vec{\eta}_+, \vec{\eta}'_-) \frac{\partial}{\partial \eta_1} |\vec{x} - \vec{\eta}_+|^{-1} \frac{\partial}{\partial \eta'_1} |\vec{x}' - \vec{\eta}'_-|^{-1} -$$

$$R_{33}(\vec{\eta}_-, \vec{\eta}'_+) \frac{\partial}{\partial \eta_1} |\vec{x} - \vec{\eta}_-|^{-1} \frac{\partial}{\partial \eta'_1} |\vec{x}' - \vec{\eta}'_+|^{-1} +$$

$$R_{33}(\vec{\eta}_-, \vec{\eta}'_-) \frac{\partial}{\partial \eta_1} |\vec{x} - \vec{\eta}_-|^{-1} \frac{\partial}{\partial \eta'_1} |\vec{x}' - \vec{\eta}'_-|^{-1} \Big\} d\vec{\eta}\, d\vec{\eta}' . \tag{4.2}$$

If we assume now that the turbulence is homogeneous, we can make changes of variables in (4.2) similar to those of Equations (2.17) - (2.19). Equation (4.2) then becomes

$$\langle p(\vec{x}) p(\vec{x}')\rangle_{TM} =$$

$$\frac{\rho^2 V^2}{4\pi^2} \Big\{ \int R_{33}(\vec{\zeta}_0) \int \frac{x_1 - \zeta_1 - \xi_1}{|\vec{x} - \vec{\zeta}_0 - \vec{\xi}_+|^3} \; \frac{x'_1 - \xi_1}{|\vec{x}' - \vec{\xi}_+|^3} \, d\vec{\xi}\, d\vec{\zeta} -$$

$$\int R_{33}(\vec{\zeta}_+) \int \frac{x_1 - \zeta_1 - \xi_1}{|\vec{x} - \vec{\zeta}_+ - \vec{\xi}_-|^3} \; \frac{x'_1 - \xi_1}{|\vec{x}' - \vec{\xi}_-|^3} \, d\vec{\xi}\, d\vec{\zeta} -$$

$$\int R_{33}(\vec{\zeta}_-) \int \frac{x_1 - \zeta_1 - \xi_1}{|\vec{x} - \vec{\zeta}_- - \vec{\xi}_+|^3} \; \frac{x'_1 - \xi_1}{|\vec{x}' - \vec{\xi}_+|^3} \, d\vec{\xi}\, d\vec{\zeta} +$$

$$\int R_{33}(\vec{\zeta}_0)\int \frac{x_1-\zeta_1-\xi_1}{|\vec{x}-\vec{\zeta}_0-\vec{\xi}_-|^3}\;\frac{x_1'-\xi_1}{|\vec{x}'-\vec{\xi}_-|^3}\,d\vec{\xi}\,d\vec{\zeta}\Bigg\}, \tag{4.3}$$

where

$$\vec{\zeta}_0 = (\zeta_1, \zeta_2, 0),$$

$$\vec{\zeta}_+ = (\zeta_1, \zeta_2, 2a),$$

$$\vec{\zeta}_- = (\zeta_1, \zeta_2, -2a),$$

$$\vec{\xi}_+ = (\xi_1, \xi_2, a),$$

$$\vec{\xi}_- = (\xi_1, \xi_2, -a),$$

$$\vec{\zeta} = (\zeta_1, \zeta_2),$$

$$\vec{\xi} = (\xi_1, \xi_2).$$

To obtain the formula for the mean square pressure at a distance h from the x_1, x_2 plane we set $x = x' = (0,0,h)$ in (4.3) with the result

$$\langle p^2(h)\rangle_{TM} =$$

$$\frac{\rho^2 V^2}{4\pi^2}\Bigg\{\int R_{33}(\vec{\zeta}_0)\int \frac{\xi_1+\zeta_1}{\left[|\vec{\xi}+\vec{\zeta}|^2+(h-a)^2\right]^{3/2}}\;\frac{\xi_1}{\left[\xi^2+(h-a)^2\right]^{3/2}}\,d\vec{\xi}\,d\vec{\zeta} \;-$$

$$\int R_{33}(\vec{\zeta}_+)\int \frac{\xi_1+\zeta_1}{\left[|\vec{\xi}+\vec{\zeta}|^2+(h-a)^2\right]^{3/2}}\;\frac{\xi_1}{\left[\xi^2+(h+a)^2\right]^{3/2}}\,d\vec{\xi}\,d\vec{\zeta} \;-$$

$$\int R_{33}(\vec{\zeta}_-)\int \frac{\xi_1+\zeta_1}{\left[|\vec{\xi}+\vec{\zeta}|^2+(h+a)^2\right]^{3/2}}\;\frac{\xi_1}{\left[\xi^2+(h-a)^2\right]^{3/2}}\,d\vec{\xi}\,d\vec{\zeta}\;+$$

$$\left.\int R_{33}(\vec{\zeta}_0)\int \frac{\xi_1+\zeta_1}{\left[|\vec{\xi}+\vec{\zeta}|^2+(h+a)^2\right]^{3/2}}\;\frac{\xi_1}{\left[\xi^2+(h+a)^2\right]^{3/2}}\,d\vec{\xi}\,d\vec{\zeta}\right\}. \tag{4.4}$$

In examining the four terms occurring on the right hand side of (4.4), it is seen that the two terms involving $\vec{\zeta}_0$ are of the same form as that of the single shear layer (Equation 2.20) and therefore give the contribution to $\langle p^2(h)\rangle_{TM}$ which would be expected if the shear layers acted independently. The two terms involving $\vec{\zeta}_+$ and $\vec{\zeta}_-$ give the contribution to $\langle p^2(h)\rangle_{TM}$ owing to the correlation between the fluctuating pressure due to one shear layer and the fluctuating pressure due to the other shear layer. We shall refer to this effect as interaction between the shear layers. Then, in general, the mean square fluctuating pressure due to the interaction of turbulence with mean shear in multiple shear layers is not a simple superposition of the contributions of the shear layers acting independently. It is, however, clear from Equation (4.4) that if the distance between the shear layers is large in comparison with the correlation length of the turbulence, then the terms in (4.4) corresponding to interaction between the shear layers can be neglected, since then $R_{33}(\vec{\zeta}_+)$ and $R_{33}(\vec{\zeta}_-)$ will be small in comparison with $R_{33}(\vec{\zeta}_0)$. In this case $\langle p^2(h)\rangle_{TM}$ can be considered as a superposition of the effects of the shear layers acting independently.

For the case in which the distance between shear layers is of the order of the correlation length of the turbulence, the four $\vec{\xi}$ integrals occurring in (4.4) can be evaluated using the formula derived in Appendix A. We shall consider here the case in which the distance from each of the shear layers to the point x is large in comparison to the distance between the shear layers, i.e., $|h| >> a$. Then we can replace $(h \pm a)^2$ by h^2 everywhere in (4.4) without introducing significant error, and, using (A16), Equation (4.4) reduces to

$$\langle p^2(h)\rangle_{TM} \cong (2\pi)^{-1}\rho^2 V^2 \int \left[2R_{33}(\vec{\zeta}_0) - R_{33}(\vec{\zeta}_+) - R_{33}(\vec{\zeta}_-)\right] \cdot$$

$$\zeta^{-2}\left\{(1 - 2\zeta_1^2/\zeta^2)\left[1-(1+\zeta^2/4h^2)^{-\frac{1}{2}}\right] + \right.$$

$$\left.(\zeta_1^2/4h^2)(1+\zeta^2/4h^2)^{-3/2}\right\} d\vec{\zeta} \,. \tag{4.5}$$

Note also that by substituting a for h on the right hand side of (4.5), which is equivalent to setting $h = 0$ in (4.4), we obtain the exact formula for the mean square fluctuating pressure at a point midway between the shear layers.

Assuming that the turbulence is isotropic we can write

$$R_{33}(\vec{\zeta}_0) = v^2\left[f(\zeta) + \frac{1}{2}\zeta f'(\zeta)\right],$$

$$R_{33}(\vec{\zeta}_+) = R_{33}(\vec{\zeta}_-) = v^2\left[f(\zeta_+) + \frac{1}{2}\zeta_+(1 - 4a^2/\zeta_+^2)f'(\zeta_+)\right],$$

so that, letting $b = 2a$ be the distance between the shear layers, (4.5) becomes

$$\langle p^2(h)\rangle_{TM} \cong \pi^{-1}\rho^2 V^2 v^2 \int \zeta^{-2}\left[f(\zeta) + \frac{1}{2}\zeta f'(\zeta)\right]\Big\{(1 - 2\zeta_1^2/\zeta^2)\Big[1 - (1 + \zeta^2/4h^2)^{-\frac{1}{2}}\Big] + (\zeta_1^2/4h^2)(1 + \zeta^2/4h^2)^{-3/2}\Big\}\,d\vec{\zeta} - \pi^{-1}\rho^2 V^2 v^2 \int \zeta^{-2}\Big[f(\zeta_+) + \frac{1}{2}\zeta_+(1 - b^2/\zeta_+^2)\,f'(\zeta_+)\Big]\Big\{(1 - 2\zeta_1^2/\zeta^2)\Big[1 - (1 + \zeta^2/4h^2)^{-\frac{1}{2}}\Big] + (\zeta_1^2/4h^2)(1 + \zeta^2/4h^2)^{-3/2}\Big\}\,d\vec{\zeta}\,. \tag{4.6}$$

The first integral on the right hand side of (4.6) is, except for a factor of two, identical to one encountered previously (see Equations (2.23)-(2.25)), and so we can write

$$\pi^{-1}\rho^2 V^2 v^2 \int \zeta^{-2}\left[f(\zeta) + \frac{1}{2}\zeta f'(\zeta)\right]\Big\{(1 - 2\zeta_1^2/\zeta^2)\Big[1 - (1 + \zeta^2/4h^2)^{-\frac{1}{2}}\Big] + (\zeta_1^2/4h^2)(1 + \zeta^2/4h^2)^{-3/2}\Big\}\,d\vec{\zeta} = \frac{3}{2}\rho^2 V^2 v^2 z^{-4}\int_0^\infty \zeta^3(1 + \zeta^2/z^2)^{-5/2} f(\zeta)\,d\zeta\,, \tag{4.7}$$

where, as before, $z = 2h$. Writing the second integral on the right hand side of (4.6) in terms of polar coordinates as in (2.23), and carrying out the integration with respect to the angular coordinate, we obtain

$$\pi^{-1}\rho^2 V^2 v^2 \int \zeta^{-2}\left[f(\zeta_+) + \frac{1}{2}\zeta_+(1 - b^2/\zeta_+^2) f'(\zeta_+)\right]\Big\{(1 - 2\zeta_1^2/\zeta^2)\Big[1 -$$

$$(1+\zeta^2/4h^2)^{-\frac{1}{2}}\Big] + (\zeta_1^2/4h^2)(1+\zeta^2/4h^2)^{-3/2}\Big\} d\vec{\zeta} =$$

$$\rho^2 V^2 v^2 z^{-2}\int_0^{\infty} \zeta\left[f(\zeta_+) + \frac{1}{2}\zeta_+(1 - b^2/\zeta_+^2) f'(\zeta_+)\right](1+\zeta^2/z^2)^{-3/2}\, d\zeta \;.$$

Making a change of variable, we can write

$$\rho^2 V^2 v^2 z^{-2}\int_0^{\infty} \zeta\left[f(\zeta_+) + \frac{1}{2}\zeta_+(1 - b^2/\zeta_+^2) f'(\zeta_+)\right](1+\zeta^2/z^2)^{-3/2}\, d\zeta =$$

$$\rho^2 V^2 v^2 z^{-2}\int_b^{\infty} \left[\zeta f(\zeta) + \frac{1}{2}(\zeta^2 - b^2) f'(\zeta)\right]\left[1 + (\zeta^2 - b^2)/z^2\right]^{-3/2} d\zeta \;.$$

Noting that

$$\zeta f(\zeta) + \frac{1}{2}(\zeta^2 - b^2) f'(\zeta) = \frac{d}{d\zeta}\left[\frac{1}{2}(\zeta^2 - b^2) f(\zeta)\right]$$

we can integrate by parts to obtain finally

$$\pi^{-1}\rho^2 V^2 v^2 \int \zeta^{-2}\left[f(\zeta_+) + \frac{1}{2}\zeta_+(1 - b^2/\zeta_+^2) f'(\zeta_+)\right]\Big\{(1 - 2\zeta_1^2/\zeta^2)\Big[1 -$$

$$(1+\zeta^2/4h^2)^{-\frac{1}{2}}\Big] + (\zeta_1^2/4h^2)(1+\zeta^2/4h^2)^{-3/2}\Big\} d\vec{\zeta} =$$

$$\frac{3}{2}\rho^2 V^2 v^2 z^{-4} \int_b^\infty (\zeta^2 - b^2)\left[1 + (\zeta^2 - b^2)/z^2\right]^{-5/2} \zeta f(\zeta)\, d\zeta \ . \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.6) we have

$$\langle p^2(h)\rangle_{TM} \cong \frac{3}{2}\rho^2 V^2 v^2 z^{-4} \left\{ \int_0^\infty \zeta^3 (1 + \zeta^2/z^2)^{-5/2} f(\zeta)\, d\zeta \ - \right.$$

$$\left. \int_b^\infty (\zeta^2 - b^2)\left[1 + (\zeta^2 - b^2)/z^2\right]^{-5/2} \zeta f(\zeta)\, d\zeta \right\}, \tag{4.9}$$

where we have assumed that $|z| >> b$. The first term on the right hand side of (4.9) is just twice the mean square fluctuating pressure for the case of a single shear layer coincident with the $x_1 - x_2$ plane, and gives the contribution to $\langle p^2(h)\rangle_{TM}$ due to the effects of the shear layers acting independently. The second term gives the contribution to $\langle p^2(h)\rangle_{TM}$ due to interaction between the shear layers. This term is negative, i.e., in this case interaction between the shear layers tends to reduce the mean square pressure fluctuations.

As noted previously the exact formula for the mean square fluctuating pressure at a point midway between the shear layer, i.e., for $h = 0$, is obtained by replacing h by a everywhere in the right hand side of (4.5), i.e.,

$$\langle p^2(0)\rangle_{TM} = (2\pi)^{-1} \rho^2 V^2 \int \left[2R_{33}(\vec{\zeta_0}) - R_{33}(\vec{\zeta_+}) - R_{33}(\vec{\zeta_-}) \right]$$

$$\left\{(1-2\zeta_1^2/\zeta^2)\left[1-(1+\zeta^2/4a^2)^{-\frac{1}{2}}\right]+(\zeta_1^2/4a^2)(1+\zeta^2/4a^2)^{-3/2}\right\}\zeta^{-2}d\vec{\zeta}. \tag{4.10}$$

For isotropic turbulence then we obtain after replacing z by b everywhere in (4 9)

$$\langle p^2(0)\rangle_{TM} = \frac{3}{2}\rho^2 V^2 v^2 \left\{ b\int_0^\infty \zeta^3(\zeta^2+b^2)^{-5/2} f(\zeta)\,d\zeta \; - \right.$$

$$\left. b\int_b^\infty \zeta^{-4}(\zeta^2-b^2)\,f(\zeta)\,d\zeta \right\}. \tag{4.11}$$

Again we see that interaction between the shear layers tends to reduce the mean square fluctuating pressure.

5.0 APPLICATION TO BOUNDARY LAYER TURBULENCE

The results of the preceding sections can be used to give rough order-of-magnitude estimates for the mean square fluctuating pressure owing to turbulence-mean shear interaction in a turbulent boundary layer. The procedure is to first approximate the flow near the boundary by means of a mirror flow model similar to the one described by Kraichnan[4]. The boundary layer is then approximated by a shear layer lying at some distance from the boundary which is less than the boundary layer thickness, and across which the mean flow velocity parallel to the boundary changes abruptly from zero to the external flow velocity. In other words, the mean shear, instead of being distributed over the boundary layer, is assumed to be concentrated in the plane of the shear layer.

We assume that the boundary is coincident with the $x_1 - x_2$ plane and construct the mirror flow model as follows: Assume $u_i(\vec{x},t)$ represents the turbulent velocity corresponding to homogeneous turbulence. Then, following Kraichnan,[4] we define new coordinates $\vec{x}^*$ by

$$x_1^* = x_1 , \; x_2^* = x_2 , \; x_3^* = -x_3 \tag{5.1}$$

and a new velocity field u_i^* (x,t) by

$$\begin{aligned} u_1^*(\vec{x},t) &= 2^{-\frac{1}{2}} \left[u_1(\vec{x},t) + u_1(\vec{x}^*,t) \right] \\ u_2^*(\vec{x},t) &= 2^{-\frac{1}{2}} \left[u_2(\vec{x},t) + u_2(\vec{x}^*,t) \right] \end{aligned} \tag{5.2}$$

$$u_3^*(\vec{x},t) = 2^{-\frac{1}{2}}\left[u_3(\vec{x},t) - u_3(\vec{x}^*,t)\right]$$

Then the velocity field represented by $u_i^*(\vec{x},t)$ has mirror symmetry about the plane $x_3 = 0$. It is easily verified that $u_i^*(\vec{x},t)$ satisfies the continuity equation

$$\frac{\partial u_i^*}{\partial x_i} = 0$$

provided $u_i(\vec{x},t)$ satisfies it, and also that $u_i^*(\vec{x},t)$ approximates the flow near the boundary in that $u_i^*(\vec{x},t)$ and all its even derivatives with respect to x_3 vanish on the plane $x_3 = 0$. It is also easily verified that the mean square values of u_i^* far from the wall tend to the values for the homogeneous field. At the wall, however, the mean square values of u_1^* and u_2^* are twice the values of the respective homogeneous components, whereas the mean square value of u_3^* at the wall is zero. (For a more detailed discussion of the mirror flow velocity field, see the above-mentioned paper of Kraichnan's.) Then, defining $R_{ij}^*(\vec{x},\vec{y})$ to be the velocity correlation for the mirror flow, we have, from (5.2).

$$R_{33}^*(\vec{x},\vec{y}) = \frac{1}{2}\quad R_{33}(\vec{x}-\vec{y}) - R_{33}(\vec{x}^*-\vec{y}) - R_{33}(\vec{x}-\vec{y}^*) + R_{33}(\vec{x}^*-\vec{y}^*) \qquad (5.3)$$

The mean flow distribution is taken to be a shear layer at $x_3 = a$, with the mean velocity zero for $0 \leq x_3 < a$, and equal to V for $x_3 > a$. Reflecting this flow in the boundary plane we obtain the case of the double shear layer identical to the one discussed in the previous section. We can then obtain the formula for the mean square fluctuating pressure owing to turbulence-mean shear interaction by substituting the mirror-flow velocity correlation given by (5.3) into Equation (4.2) of the previous section. Doing this, and noting that $\vec{\eta}_+^* = \vec{\eta}_-$, $\vec{\eta}_-^* = \vec{\eta}_+$, and $R_{33}^*(\vec{\eta}_+, \vec{\eta}_+') = R_{33}^*(\vec{\eta}_-, \vec{\eta}_-') = -R_{33}^*(\vec{\eta}_+, \vec{\eta}_-') = -R_{33}^*(\vec{\eta}_-, \vec{\eta}_+')$, we obtain

$$\langle p(\vec{x})p(\vec{x}\,')\rangle_{TM} = \frac{\rho^2 V^2}{8\pi^2}\iint \Big[R_{33}(\vec{\eta}_+ - \vec{\eta}_+') - R_{33}(\vec{\eta}_- - \vec{\eta}_+') - R_{33}(\vec{\eta}_+ - \vec{\eta}_-') + R_{33}(\vec{\eta}_- - \vec{\eta}_-')\Big]\Big[\frac{\partial}{\partial x_1}|\vec{x} - \vec{\eta}_+|^{-1}\frac{\partial}{\partial x_1'}|\vec{x}\,' - \vec{\eta}_+'|^{-1} + \frac{\partial}{\partial x_1}|\vec{x} - \vec{\eta}_-|^{-1}\frac{\partial}{\partial x_1'}|\vec{x}\,' - \vec{\eta}_-'|^{-1} + \frac{\partial}{\partial x_1}|\vec{x} - \vec{\eta}_+|^{-1}\frac{\partial}{\partial x_1'}|\vec{x}\,' - \vec{\eta}_-'|^{-1} + \frac{\partial}{\partial x_1}|\vec{x} - \vec{\eta}_-|^{-1}\frac{\partial}{\partial x_1'}|\vec{x}\,' - \vec{\eta}_+'|^{-1}\Big] d\vec{\eta}\, d\vec{\eta}\,' . \qquad (5.4)$$

Making changes of variables similar to those of Equations (2.17)-(2.19), (5.4) becomes

$$\langle p(\vec{x})p(\vec{x}\,')\rangle_{TM} = \frac{\rho^2 V^2}{8\pi^2}\int \Big[2R_{33}(\vec{\zeta}_0) - R_{33}(\vec{\zeta}_+) - R_{33}(\vec{\zeta}_-)\Big] \cdot \left\{\iint \frac{x_1 - \zeta_1 - \xi_1}{|\vec{x} - \vec{\zeta}_0 - \vec{\xi}_+|^3}\,\frac{x_1' - \xi_1}{|\vec{x}\,' - \vec{\xi}_+|^3}\, d\vec{\xi} + \right.$$

$$\int \frac{x_1 - \zeta_1 - \xi_1}{\left|\vec{x} - \vec{\zeta}_0 - \vec{\xi}_-\right|^3} \quad \frac{x_1' - \xi_1}{\left|\vec{x}' - \vec{\xi}_-\right|^3} \, d\vec{\xi} \; +$$

$$\int \frac{x_1 - \zeta_1 - \xi_1}{\left|\vec{x} - \vec{\zeta}_+ - \vec{\xi}_-\right|^3} \quad \frac{x_1' - \xi_1}{\left|\vec{x}' - \vec{\xi}_-\right|^3} \, d\vec{\xi} \; +$$

$$\left. \int \frac{x_1 - \zeta_1 - \xi_1}{\left|\vec{x} - \vec{\zeta}_- - \vec{\xi}_+\right|^3} \quad \frac{x_1' - \xi_1}{\left|\vec{x}' - \vec{\xi}_+\right|^3} \, d\vec{\xi} \right\} d\vec{\zeta}, \qquad (5.5)$$

where $\vec{\zeta}_0$, $\vec{\zeta}_+$, $\vec{\zeta}_-$, $\vec{\zeta}$, $\vec{\xi}$, $\vec{\xi}_+$, $\vec{\xi}_-$ are as defined previously. Setting $\vec{x} = \vec{x}' = (0,0,h)$ in (5.5) we obtain the formula for the mean square fluctuating pressure owing to turbulence-mean shear interaction at a distance h from the boundary:

$$\langle p^2(h) \rangle_{TM} = \frac{\rho^2 V^2}{8\pi^2} \int \left[2 R_{33}(\vec{\zeta}_0) - R_{33}(\vec{\zeta}_+) - R_{33}(\vec{\zeta}_-) \right] \cdot$$

$$\left\{ \int \frac{\xi_1 + \zeta_1}{\left[\left|\vec{\xi} + \vec{\zeta}\right|^2 + (h-a)^2 \right]^{3/2}} \quad \frac{\xi_1}{\left[\xi^2 + (h-a)^2 \right]^{3/2}} \, d\vec{\xi} \; + \right.$$

$$\int \frac{\xi_1 + \zeta_1}{\left[\left|\vec{\xi} + \vec{\zeta}\right|^2 + (h+a)^2 \right]^{3/2}} \quad \frac{\xi_1}{\left[\xi^2 + (h+a)^2 \right]^{3/2}} \, d\vec{\xi} \; +$$

$$\int \frac{\xi_1 + \zeta_1}{\left[\left|\vec{\xi} + \vec{\zeta}\right|^2 + (h - a)^2\right]^{3/2}} \; \frac{\xi_1}{\left[\xi^2 + (h + a)^2\right]^{3/2}} \, d\vec{\xi} +$$

$$\left. \int \frac{\xi_1 + \zeta_1}{\left[\left|\vec{\xi} + \vec{\zeta}\right|^2 + (h + a)^2\right]^{3/2}} \; \frac{\xi_1}{\left[\xi^2 + (h - a)^2\right]^{3/2}} \, d\vec{\xi} \right\} d\vec{\zeta}. \tag{5.6}$$

Setting $h = 0$ in (5.6), and using (A16), we obtain the formula for the mean square fluctuating pressure at the boundary owing to turbulence-mean shear interaction, which we denote by $\langle p_w^2 \rangle_{TM}$:

$$\begin{aligned} \langle p_w^2 \rangle_{TM} = \pi^{-1} \rho^2 V^2 \int \left[2R_{33}(\vec{\zeta}_0) - R_{33}(\vec{\zeta}_+) - R_{33}(\vec{\zeta}_-) \right] \cdot \\ \left\{ (1 - 2\zeta_1^2/\zeta^2) \left[1 - (1 + \zeta^2/b^2)^{-\frac{1}{2}} \right] + \right. \\ \left. (\zeta_1^2/b^2)(1 + \zeta^2/b^2)^{-3/2} \right\} \zeta^{-2} \, d\vec{\zeta} \, , \end{aligned} \tag{5.7}$$

where $b = 2a$. The value of $\langle p_w^2 \rangle_{TM}$ given by (5.7) is just twice the mean square fluctuating pressure at a point midway between the two shear layers for the case of homogeneous turbulence, as given by formula (4.10). Then for the case in which R_{33} in (5.7) corresponds to isotropic turbulence, $\langle p_w^2 \rangle_{TM}$ is equal to twice the value of $\langle p^2(0) \rangle_{TM}$ given by (4.11), i.e.,

$$\langle p_w^2 \rangle_{TM} = 3\rho^2 V^2 v^2 \left\{ b \int_0^\infty \zeta^3 (\zeta^2 + b^2)^{-5/2} f(\zeta) \, d\zeta \, - \right.$$

$$b\int_{b}^{\infty} \zeta^{-4}(\zeta^2 - b^2) f(\zeta)\, d\zeta \Bigg\} , \qquad (5.8)$$

where v is defined as before.

In order to obtain estimates for $\langle p_w^2 \rangle_{TM}$ for an actual boundary layer flow it is necessary to relate the parameters appearing in (5.8), i.e., V, v, b and the correlation length σ^{-1} of the turbulence, to the boundary layer flow parameters; namely τ_0, the wall shear stress, and U_τ the friction velocity, where $U_\tau^2 = \tau_0/\rho$. We first assume that the mean velocity distribution in the boundary layer is of the form

$$U_1(x_3) = V(1 - e^{-\beta x_3}) . \qquad (5.9)$$

where V is the external flow velocity. Differentiating (5.9) we obtain

$$S(x_3) = S_0 e^{-\beta x_3} , \qquad (5.10)$$

where $S(x_3)$ is the mean shear, and $S_0 = \beta V$ is the mean shear at the wall. We now choose a, the distance from the wall to the shear layer, to be one-third the distance from the wall to the point at which the mean velocity is approximately 95 percent of the external flow velocity. Accordingly, by (5.9), we choose $\beta a = 1$, or $\beta b = 2$. Kraichnan[4] states that (5.10) gives a reasonable approximation to the mean shear in an actual turbulent boundary layer, at least in that region outside the laminar sublayer which is expected to give the major contribution to the pressure fluctuations, if S_0 and β are chosen so that $S_0 = 12U_\tau \sigma$ and $\beta = \sigma$. Also, a reasonable value of v, according to experimental evidence, is $1.5U_\tau$. Then, since $S_0 = \beta V$, we have

$$V = 12U_\tau, \; \sigma b = 2, \; v = 1.5U_\tau . \qquad (5.11)$$

Substituting (5.11) into (5.8) we obtain

$$\left[\langle p_w^2\rangle_{TM}\right]^{\frac{1}{2}} \cong 18\sqrt{3}\,\rho U_\tau^2\left[b\int_0^\infty \zeta^3(\zeta^2+b^2)^{-5/2} f(\zeta)d\zeta \; - \right.$$

$$\left. b\int_b^\infty \zeta^{-4}(\zeta^2-b^2)f(\zeta)\,d\zeta\right]^{\frac{1}{2}} . \qquad (5.12)$$

Since b is twice the correlation length of the turbulence, the second integral on the right hand side of (5.12) can be neglected. Then taking $f(\zeta) = e^{-\sigma^2\zeta^2}$, and setting $\sigma b = 2$, (5.12) becomes, after a change of variable,

$$\left[\langle p_w^2\rangle_{TM}\right]^{\frac{1}{2}} \cong 18\sqrt{3}\,\rho\, U_\tau^2\left[\int_0^\infty \eta^3(1+\eta^2)^{-5/2}\, e^{-4\eta^2}\, d\eta\right]^{\frac{1}{2}} . \qquad (5.13)$$

The integral occurring on the right hand side of (5.13) can be evaluated approximately by various methods. One way is to first observe that

$$\int_0^\infty \eta^3(1+\eta^2)^{-5/2}\, e^{-4\eta^2}\, d\eta \cong \int_0^1 \eta^3(1+\eta^2)^{-5/2}\, e^{-4\eta^2}\, d\eta .$$

We can then approximate $e^{-4\eta^2}$ in the interval $0 \leq \eta \leq 1$ by the function $1 - 2\eta^2 + \eta^4$. The resulting integral can be evaluated by means of tables to yield finally

$$\left[\left\langle p_w^2 \right\rangle_{TM}\right]^{\frac{1}{2}} \cong 4.4\, \rho U_\tau^2 = 4.4\, \tau_0 \, . \tag{5.14}$$

In view of the many assumptions and approximations made in the above analysis, Equation (5.14) should be considered only a crude, order-of-magnitude estimate of the root mean square fluctuating wall pressure. However, it is of interest to note that the numerical factor 4.4 appearing in Equation (5.14) agrees reasonably well with other published results; e.g., Kraichnan[4] given a numerical factor of 6, while results obtained by Lilley[5] indicate a numerical factor of 3.1. Experimental results of Bull[7] indicate the numerical factor generally lies between 2 and 3.

6.0 CONCLUDING REMARKS

Although in the preceding sections emphasis was placed on finding the mean square fluctuating pressure $\langle p^2 \rangle_{IM}$, the results obtained can also be used to calculate the two-point pressure correlation under various conditions; e.g. formula (2.17) gives the two point pressure correlation function owing to the interaction of (not necessarily homogeneous) turbulence with a single shear layer, while (2.19) gives the same result for homogeneous turbulence. Analogous results (formulas (4.2) and (4.3)) are obtained for the double shear layer. The case of homogeneous turbulence with scale anisotropy can be included by substituting the expression for the velocity correlation given by Equation (3.1) into (2.19) and (4.3). It should also be noted that since the analysis given here does not involve the time variable explicitly, the effect of time difference can be included in the above-mentioned pressure correlations simply by assuming that the velocity correlation R_{ij} appearing on the right hand side of the above-mentioned equations is also a function of the time difference.

As was seen in Section 3, departure from isotropy results in a decrease in the mean square fluctuating pressure in the near field of a single shear layer, whereas in the far field, anisotropy results in an increase in mean square fluctuating pressure. In view of Kraichnan's results for the interaction of turbulence with uniform mean shear[3], the near-field effects of anisotropy are not unexpected; however, the physical interpretation of the difference in near-field and far-field effects of anisotropy obtained here remains unclear.

Results obtained in Section 4 for the case of the double shear layer indicate that, when the separation between the shear layers is of the order of the correlation length of the turbulence, interaction between the shear layers (i.e., the effect of correlation between the fluctuating pressure due to one shear layer and the fluctuating pressure due to the other shear layer) is significant and results in a decrease in the mean square fluctuating pressure compared to the value which would be expected if the shear layers acted independently. This decrease is the result of the particular velocity profile chosen in the [illegible] rather than being an intrinsic characteristic of multiple shear layers. The configuration chosen has mirror symmetry about the $x_1 - x_2$ plane; i.e., $U_1(-x_3) = U_1(x_3)$, so that the mean shear is anti-symmetric about this plane. Thus, in the interaction term in the expression for the mean square fluctuating pressure or pressure correlation (see (2.13)), the product of the mean shear at a point on one shear layer with the mean shear at a point on the other shear layer is negative. If the configuration is chosen with the mean shear having anti-symmetry about the $x_1 - x_2$ plane, so that the mean shear is symmetric, then interaction between the shear layers will result in an increase in mean square fluctuating pressure.

Order-of-magnitude estimates obtained in Section 5 for the mean square fluctuating wall pressure in a turbulent boundary layer show reasonably good agreement with other published results. Although this agreement is to a certain extent fortuitous, in that the final result is dependent upon certain assumed parameters such as the distance from the wall to the shear layer, it is nonetheless encouraging since it indicates that refinements of the method given in the text might well result in even closer agreement with other results, both theoretical and experimental. Such refinements might include an attempt to approximate the mean velocity profile in the boundary layer by several shear layers, instead of one as was done in the text. (It is clear, in fact, since any piecewise continuous function can be approximated arbitrarily closely by means of step functions, that any mean velocity profile can be approximated as closely as we please by a finite number of shear layers. More complicated velocity profiles, such as the one which exists near the exit of a jet, might be approximated using a combination of shear layers and uniform mean shear; e.g., the velocity profile

$$U_1(x_3) = V_1 + \alpha x_3; \quad x_3 < 0$$

$$U_1(x_3) = V_2 + \alpha x_3; \quad x_3 > 0$$

where $V_1 \neq V_2$, gives a shear layer at the plane $x_3 = 0$ superimposed on uniform mean shear α.) It should be noted, however, that the degree of computational difficulty will increase roughly as the square of the number of shear layers chosen, since interaction between the shear layers must, in general, be considered.

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APPENDIX

It is our objective here to evaluate the integral

$$I_0 = \int \frac{\xi_1}{|\vec{\xi} - \vec{x}|^3} \; \frac{\xi_1 + \zeta_1}{|\vec{\xi} - (\vec{x} - \vec{\zeta})|^3} \; d\xi_1 d\xi_2 \quad , \tag{A1}$$

where $\vec{x} = (0,0,h)$, $\vec{\xi} = (\xi_1, \xi_2, 0)$, $\vec{\zeta} = (\zeta_1, \zeta_2, 0)$. Equation (A1) can be written in the alternate form

$$I_0 = \int \frac{\xi_1}{(h^2 + \xi^2)^{3/2}} \quad \frac{\xi_1 + \zeta_1}{(h^2 + |\vec{\xi} + \vec{\zeta}|^2)^{3/2}} \; d\vec{\xi} \quad , \tag{A2}$$

where $\vec{\xi} = (\xi_1, \xi_2)$, $\vec{\zeta} = (\zeta_1, \zeta_2)$. It is convenient to consider a somewhat more general form of (A2); namely

$$I(\vec{\eta}, \vec{\zeta}; a, b) = \int \frac{\xi_1 + \eta_1}{(b^2 + |\vec{\xi} + \vec{\eta}|^2)^{3/2}} \quad \frac{\xi_1 + \zeta_1}{(a^2 + |\vec{\xi} + \vec{\zeta}|^2)^{3/3}} \; d\vec{\xi} \quad , \tag{A3}$$

where $\vec{\eta} = (\eta_1, \eta_2)$, and $a, b > 0$. Then

$$I_0 = I(0, \vec{\zeta}; h, h) \quad . \tag{A4}$$

We now proceed to evaluate $I(\vec{\eta}, \vec{\zeta}; a, b)$ using a method similar to that given by Kraichnan[3], which is based on a method of Feynman[9]. We first write (A3) in the form

$$I(\vec{\eta}, \vec{\zeta}; a, b) = \int \frac{\partial^2}{\partial \eta_1 \partial \zeta_1} \left(\frac{1}{r r'} \right) d\vec{\xi} \, , \tag{A5}$$

where $r = (b^2 + |\vec{\xi} + \vec{\eta}|^2)^{\frac{1}{2}}$, $r' = (a^2 + |\vec{\xi} + \vec{\zeta}|^2)^{3/2}$. Now

$$\int \frac{\partial^2}{\partial\eta_1 \partial\zeta_1}\left(\frac{1}{rr'}\right) d\vec{\xi} = \lim_{A\to\infty} \int_A \frac{\partial^2}{\partial\eta_1\partial\zeta_1}\left(\frac{1}{rr'}\right) d\vec{\xi} ,$$

where A is any finite area in (ξ_1, ξ_2) space, which in the limit as $A \to \infty$ is to include all of (ξ_1, ξ_2) space. Then, setting

$$I_A(\vec{\eta}, \vec{\zeta}) = \int_A \frac{1}{rr'} \, d\vec{\xi} , \tag{A6}$$

we have

$$I(\vec{\eta}, \vec{\zeta}; a, b) = \lim_{A\to\infty} \frac{\partial^2 I_A}{\partial\eta_1 \partial\zeta_1} . \tag{A7}$$

Making the change of variable $\vec{u} = \vec{\xi} + \vec{\eta}$, and setting $\vec{z} = \vec{\eta} - \vec{\zeta}$, (A6) becomes

$$I_A(\vec{\eta}, \vec{\zeta}) = \int_{A'} \frac{1}{rr'} \, d\vec{u} , \tag{A8}$$

where $r = (b^2 + u^2)^{\frac{1}{2}}$, $r' = (a^2 + |\vec{u} - \vec{z}|^2)^{\frac{1}{2}}$, and $A' = A + \vec{\eta}$. Then setting $I'_A(\vec{z}) = I_A(\vec{\eta}, \vec{\zeta})$ we have

$$I(\vec{\eta}, \vec{\zeta}; a, b) = - \lim_{A'\to\infty} \frac{\partial^2}{\partial z_1^2} I'_A(\vec{z}) , \tag{A9}$$

where

$$I'_A(\vec{z}) = \int_{A'} \frac{1}{rr'} \, d\vec{u} \; .$$

Utilizing the identity

$$\frac{1}{rr'} = \frac{2}{\pi}\int_0^{\infty} \frac{d\tau}{r'^2 + r^2\tau^2} \; ,$$

we can write

$$I'_A(\vec{z}) = \frac{2}{\pi}\int_{A'}\int_0^{\infty} \frac{d\tau}{r'^2 + r^2\tau^2} \, d\vec{u} \; . \tag{A10}$$

Substituting for r and r', and interchanging the order of integration, (A10) can be written

$$I'_A(\vec{z}) = \frac{2}{\pi}\int_0^{\infty} \frac{d\tau}{1+\tau^2}\int_{A'} \frac{d\vec{u}}{\left|\vec{u} - \frac{\vec{z}}{1+\tau^2}\right|^2 + \frac{a^2 + b^2\tau^2}{1+\tau^2} + \frac{\tau^2}{(1+\tau^2)^2} \; z^2} \; . \tag{A11}$$

If we now choose A' to be a circle of radius R centered at $\vec{z}/(1+\tau^2)$ we can carry cut the integration over A' tc obtain

$$I'_A(\vec{z}) = 2\int_0^{\infty}\left[\log R^2 + \log(1 + C^2/R^2) - \log C^2\right]\frac{d\tau}{1+\tau^2} \; , \tag{A12}$$

where $C^2 = \dfrac{a^2+b^2\tau^2}{1+\tau^2} + \dfrac{\tau^2}{(1+\tau^2)^2}\, z^2$. Now the first term on the right hand side of (A12) will give no contribution to $I(\vec{\eta}, \vec{\zeta}\,; a, b)$ since its derivative with respect to z_1 is zero. Also, the second term on the right hand side of (A12) will not contribute to $I(\vec{\eta}, \vec{\zeta}\,; a, b)$ since it vanishes as $R \to \infty$. Equation (A9) then can be written

$$I(\vec{\eta},\vec{\zeta}\,; a, b) = 2\,\frac{\partial^2}{\partial z_1^2}\int_0^\infty \log\left(\frac{a^2+b^2\tau^2}{1+\tau^2} + \frac{\tau^2}{(1+\tau^2)^2}\, z^2\right)\frac{d\tau}{1+\tau^2}\,. \quad \text{(A13)}$$

The right hand side of (A13) can be evaluated as follows: First differentiate once under the integral sign with respect to z_1 to obtain

$$I(\vec{\eta},\vec{\zeta}\,; a, b) = 4\,\frac{\partial}{\partial z_1} z_1 \int_0^\infty \frac{\tau^2}{(1+\tau^2)(a^2+b^2\tau^2)+\tau^2 z^2}\;\frac{d\tau}{1+\tau^2}\,. \qquad \text{(A14)}$$

The integral on the right hand side of (A14) can then be evaluated by means of contour integration to yield

$$\int_0^\infty \frac{\tau^2}{(1+\tau^2)(a^2+b^2\tau^2)+\tau^2 z^2}\;\frac{d\tau}{1+\tau^2} =$$

$$(\pi/2z^2)\left\{1-(a+b)\left[(a-b)^2+z^2\right]^{\frac{1}{2}}\left[(a^2+b^2+z^2)^2-4a^2b^2\right]^{-\frac{1}{2}}\right\},$$

so that, differentiating again with respect to z_1, (A14) becomes

$$I(\vec{\eta}, \vec{\zeta}; a, b) = 2\pi z^{-2}(1 - 2z_1^2/z^2)\Big\{1 -$$

$$(a + b)\left[(a - b)^2 + z^2\right]^{\frac{1}{2}}\left[(a^2 + b^2 + z^2)^2 - 4a^2b^2\right]^{-\frac{1}{2}}\Big\} -$$

$$2\pi(a + b)\, z_1^2 z^{-2}\Big\{\left[(a - b)^2 + z^2\right]^{-\frac{1}{2}}\left[(a^2 + b^2 + z^2)^2 - 4a^2b^2\right]^{-\frac{1}{2}} -$$

$$2(a^2 + b^2 + z^2)\left[(a - b)^2 + z^2\right]^{\frac{1}{2}}\left[(a^2 + b^2 + z^2)^2 - 4a^2b^2\right]^{-3/2}\Big\}, \tag{A15}$$

where $\vec{z} = \vec{\eta} - \vec{\zeta}$. This is the desired expression for $I(\vec{\eta}, \vec{\zeta}; a, b)$. Then setting $a = b = h$ and $\vec{\eta} = 0$ in (A15), we have, from (A4), the desired expression for I_0:

$$I_0 = 2\pi\zeta^{-2}\Big\{(1 - 2\zeta_1^2/\zeta^2)\left[1 - (1 + \zeta^2/4h^2)^{-\frac{1}{2}}\right] +$$

$$(\zeta_1^2/4h^2)(1 + \zeta^2/4h^2)^{-3/2}\Big\}. \tag{A16}$$